International Journal of Theoretical Physics, Vol. 37, No. 5, 1998

Frequency Shift of Spectral Lines Generated by Multiple Dynamic Scattering

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Received September 1, 1997

The frequency shift of spectral lines generated by single dynamic scattering is known from Wolf's pioneering work. The multiple scattering effect is discussed and for the case of redshift two main results are shown: (1) the multiple scattering effect causes a larger shift than does single scattering, and (2) m scatterings with scattering angle θ/m each produce a larger shift than that done by n scatterings with scattering angle θ/n each if $m > n$. These results might be of particular interest in connection with the observed spectra of quasars.

1. INTRODUCTION

In the last decade some closely related processes have been discovered that can generate frequency shifts of spectral lines (Wolf, 1986, 1987; Morris and Faklis, 1987; Wolf and Foley, 1989; Wolf *et al.*, 1989; Wolf and James, 1990). One of them is scattering through random media, the study of which has been enriched greatly with the techniques of statistical optics (Wolf and Born 1980; Goodman, 1985). In scattering processes the spectral changes are induced by correlations between fluctuating response functions of the scattering medium; e.g., its dielectric susceptibility either at different points in the scatterer, when the frequency-dependent macroscopic response is time independent, or at different space-time points, when it is time dependent. In all these cases the changes in the spectrum are consequences of the correlations involving an appropriate variable characterizing the source, the field, or the response of a scatterer. With appropriate correlations the changes are manifested as frequency shifts of spectral lines. Scattering of polychromatic

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light by a medium whose dielectric susceptibility is a random function of position and time produces a frequency shift in its spectrum. This was first shown by Wolf (Wolf and Foley, 1989; Wolf *et al.*, 1989; Wolf and James, 1990). He has established that if the spectrum of the incident light consists of a single line of Gaussian profile and the correlation function of the dielectric susceptibility is also a Gaussian function, the spectrum of the scattered field will consist of a line that has approximately a Gaussian profile. However, this line is shifted toward shorter or longer wavelength, depending on the angle of scattering. For almost zero scattering angle, i.e., at about 10^{-12} *rad*, the redshift obtained from the single scattering effect is negligible, but it contributes considerably in the case of multiple scattering. Light comes to us from distant sources traveling several thousand light-years and therefore the number of scatterings on the way is expected to be large enough to produce such a redshift. This mechanism claims more accuracy in the measurment of the distance of a light source which has been overestimated by the present-day theory.

2. SCATTERING THEORY AND WOLF MECHANISM

We begin by describing briefly the main results given by Wolf and James (1990) with a little generalization by taking polychromatic instead of monochromatic light. Suppose that a polychromatic field of central frequency ω_0 is incident in the direction specified by the unit vector $\hat{u} = (u_x, u_y, u_z)$ on a scattering medium. The incident spectrum then has the form

$$S_U(\omega) = A_0\, exp\left[-\frac{1}{2\delta_0^2}(\omega - \omega_0)^2\right] \tag{1}$$

The spectrum of the scattered radiation at a point $r\hat{u}'$ in the far zone produced when a linearly polarized polychromatic plane electromagnetic wave is incident on such a medium was shown to be given by the formula (Wolf and Foley, 1989), valid within the first-order Born approximation (Wolf and Born, 1998, Goodman, 1985)

$$S^{(\infty)}(\omega') = A'^4 \int_{-\infty}^{\infty} \mathcal{K}(\omega', \omega) S_U(\omega)\, d\omega \tag{2}$$

Here $A = (2\pi)^3 V(\sin^2\psi)/c^4 r^2$, ψ being the angle between the electric vector of the incident field and $\hat{u}' = (u'_x, u'_y, u'_z$ is the unit vector in the direction of scattering; V is the volume of the scatterer, c is the speed of light, $\mathcal{K}$ is the scattering kernel defined as (Wolf and Foley, 1989)

$$\mathcal{K}(\omega', \omega) = G\left(\frac{\omega'\hat{u}' - \omega\hat{u}}{c}, \omega' - \omega; \omega\right) \tag{3}$$

where $\bar{G}(\bar{K}, \Omega, \omega)$ is the four-dimensional Fourier transform of the correlation function

$$G(\bar{R}, T; \omega) = \langle \eta^*(\bar{r} + \bar{R}, t + T; \omega)\eta(\bar{r}, t; \omega)\rangle \tag{4}$$

of the generalized dielectric susceptibility $\eta(\bar{r}, t; \omega)$ of the scattering medium. Now we study a particular case supposing that the correlation properties of the fluctuating medium are characterized by an anisotropic Gaussian function, viz.,

$$G(\bar{R}, T; \omega) = G_0\, exp\left[-\frac{1}{2}\left(\frac{X^2}{\sigma_x^2} + \frac{Y^2}{\sigma_y^2} + \frac{Z^2}{\sigma_z^2} + \frac{c^2T^2}{\sigma_\tau^2}\right)\right] \tag{5}$$

Here (X, Y, Z) are components of the vector $\bar{R}$ with respect to a suitably chosen Cartesian referrence frame; σ_x, σ_y, σ_z, σ_τ are correlation lengths and G_0 is a positive constant. The Fourier transform of $G(R, T; \omega)$ is given by

$$\begin{aligned}\bar{G}(\bar{K}, \Omega; \omega) &= \frac{1}{(2\pi)^4}\int_V d^3 R \int_{\infty}^{\infty} dT\, G(\bar{R}, T; \omega)\exp[-i(\bar{K}\cdot\bar{R} - \Omega T)] \\ &= B\exp\left[-\frac{1}{2}\left(\sigma_x^2K_x^2 + \sigma_y^2K_y^2 + \sigma_z^2\, K_z^2 + \frac{\sigma_\tau^2\Omega^2}{c^2}\right)\right]\end{aligned} \tag{6}$$

where

$$B = \frac{G_0\sigma_z\sigma_x\sigma_z\sigma_\tau}{c(2\pi)^2}$$

and $\bar{K} = (K_x, K_y, K_z)$ with the same reference frame as that of $\bar{R}$. Using (3), we get

$$\mathcal{K}(\omega', \omega) = B\exp\left[-\frac{1}{2}(\alpha'\omega'^2 - 2\beta\omega\omega' + \alpha\omega^2)\right] \tag{7}$$

where

$$\begin{aligned}\alpha &= \frac{\sigma_x^2}{c^2}u_x^2 + \frac{\sigma_y^2}{c^2}u_y^2 + \frac{\sigma_z^2}{c^2}u_z^2 + \frac{\sigma_\tau^2}{c^2} \\ \alpha' &= \frac{\sigma_x^2}{c^2}u_x'^2 + \frac{\sigma_y^2}{c^2}u_y'^2 + \frac{\sigma_z^2}{c^2}u_z'^2 + \frac{\sigma_\tau^2}{c^2}\end{aligned} \tag{8}$$

$$\beta = \frac{\sigma_x^2}{c^2} u_x u'_x + \frac{\sigma_y^2}{c^2} u_y u'_y + \frac{\sigma_z^2}{c^2} u_z u'_z + \frac{\sigma_\tau^2}{c^2}$$

According to Schwarz's inequality

$$\alpha\alpha' \geq \beta^2$$

The equality applies only when $\hat{u} \parallel \hat{u}'$.

Performing a straightforward calculation, we obtain from equations (2) and (7),

$$S^{(\infty)}(\omega') = A' \exp\left[-\frac{1}{2\delta_0'^2}(\omega' - \bar{\omega}_0)^2\right] \tag{9}$$

where

$$\begin{aligned} \bar{\omega}_0 &= \frac{|\beta|\omega_0}{\alpha' + \delta_0'^2(\alpha\alpha' - \beta^2)} \\ \delta_0'^2 &= \frac{\alpha\delta_0^2 + 1}{\alpha' + \delta_0^2(\alpha\alpha' - \beta^2)} \\ A' &= \sqrt{\frac{\pi}{2(\alpha\delta_0'^2 + 1)}}\, ABA_0\omega_0'^4\delta_0 \exp\left[\frac{|\beta|\omega_0\bar{\omega}_0 - \alpha\omega_0^2}{2(\alpha\delta_0^2 + 1)}\right] \end{aligned} \tag{10}$$

To a good approximation (Wolf and James, 1990), we can replace ω' in A' by $\bar{\omega}_0$ defined in (10), so that we can consider A' as a constant. Thus equation (9) suggests that $S^{(\infty)}(\omega')$ also has the form of a spectral line of Gaussian profile, with central frequency $\bar{\omega}_0$. The relative frequency shift is defined to be a z-number by the relation

$$z = \frac{\omega_0 - \bar{\omega}_0}{\bar{\omega}_0}$$

where ω_0 and $\bar{\omega}_0$ denote the unshifted and shifted frequency, respectively. In the above case

$$z = \frac{\alpha' + \delta_0'^2(\alpha\alpha' - \beta^2)}{|\beta|} - 1$$

Thus we see that the relative frequency shift z induced by this mechanism is independent of frequency and can take values in the range $z > -1$ even though the source, the medium, and the observer are at rest with respect to one another. It is therefore necessary to consider these effects, which can make a large contribution to the redshift of the observed spectra. Thus this mechanism of a redshift without considering a Doppler origin might play a significant role in testing cosmological models.

3. MULTIPLE SCATTERING THEORY

As we saw in the previous section, the light from a distant source is deflected on its way by a scatterer. Now the question is how these scatterers are placed in the interstellar medium. There are three different cases: (1) a continuum, (2) a stratified medium, or (3) a medium with a discrete distribution of scatterers. Here we consider the last case, *i.e.*, we assume that there are a large number of scattering volumes in the interstellar medium, distributed in a random manner. A light ray, while passing through this medium, is scattered by some of the scatterers. Let the spectrum of the source be represented by $S_0(\omega)$ and the spectrum after n scatterings by $S_n(\omega)$. Let the corresponding z-numbers be denoted by z_n, $n = 0, 1, 2, \ldots$. If the central frequency of $S_n(\omega)$ is denoted by ω_n, $n = 0, 1, 2, \ldots$, then by definition,

$$z_{n+1} = \frac{\omega_n - \omega_{n+1}}{\omega_{n+1}}, \qquad n = 0, 1, 2, \ldots$$

or

$$\frac{\omega_n}{\omega_{n+1}} = 1 + z_{n+1}, \qquad n = 0, 1, 2, \ldots \tag{11}$$

Let us find the z-number if the total number of scatterings is N. Taking the product over n from $n = 0$ to $n = N - 1$, we get

$$\frac{\omega_0}{\omega_N} = (1 + z_1)(1 + z_2) \cdots (1 + z_N) \tag{12}$$

But the left-hand side is nothing but the ratio of the source frequency and the final (i.e., after N scatterings) frequency. Hence, by definition, the final z-number z_f is given by

$$\frac{\omega_0}{\omega_N} = 1 + z_f$$

or

$$z_f = (1 + z_1)(1 + z_2) \cdots (1 + z_N) - 1 \tag{13}$$

This formula is equally valid for z-numbers which are not caused by scattering processes, i.e., (13) holds for z-numbers due to the Doppler effect, gravitational effect, and others, and thus it is a general rule for combining successive z-numbers.

3.1. Particular Case.

If we take each $z_i = z$, then the above equation reduces to

$$z_f = (1 + z)^N - 1 \tag{14}$$

It therefore asserts that for redshifts, i.e., for $z > 0$, $z_f \to \infty$ as $N \to \infty$. As light travels light-years to reach the earth, it is reasonable to assume that N depends on the distance between the earth and the source and obviously this distance is less than the measured distance obtained from present-day theory. If this relation is linear, i.e., N/D is constant, k say, which is to be measured in the unit *per light-year*, the above equation gives a possible estimate of D as follows:

$$D = \frac{1}{k} \frac{\ln(1 + z_{\text{scat}})}{\ln(1 + z)}$$

As there is no distinction between z_{scat} and the observed z-number, i.e., as we cannot distinguish the contributions in z due to different causes, we should try to find the relationship of D with the observed z-number. However, an upper bound of the contribution due to multiple scattering can be formulated in terms of the maximum total angle of scattering, as we have done previously, the maximum being taken in the sense that the image remains stellar below this limit. If we denote by z_{other} the z-number due to nonscattering processes, we get

$$1 + z_{\text{observed}} = (1 + z_{\text{other}})(1 + z_{\text{scat}})$$

Combining these results, we can get a correction for D.

3.2. Two Main Results Concerning Multiple Scattering

1. Since

$$(1 + z_f) = (1 + z_1)(1 + z_2)(1 + z_3) \cdots (1 + z_N)$$

where the z_i' and z_f are positive in the case of redshift,

$$z_f > z_1 + z_2 + z_3 + \ldots + z_N$$

and so $z_f > z_i \; \forall i = 1, 2, \ldots, N$.

Hence multiple scattering produces greater redshift than does single scattering. Figure 1 shows an example through the z–θ relation, comparing two shifts, one due to single scattering and the other to multiple scattering.

2. The effect of m scatterings with scattering angle θ/m each causes a shift given by

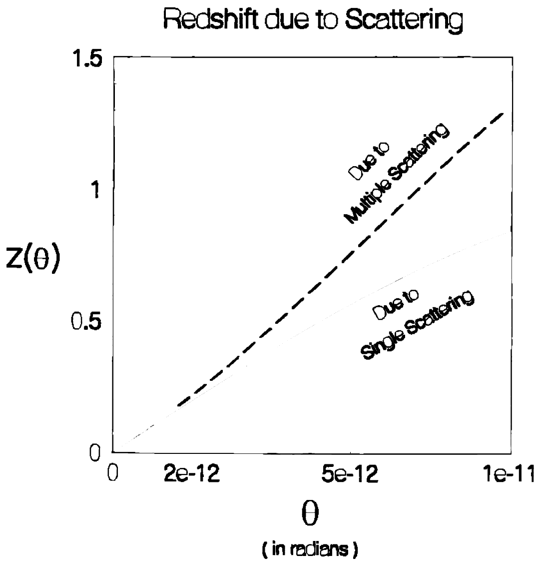

Fig. 1. Redshift due to scattering.

$$1 + z_f(m) = (1 + z_m)^m$$

where $z_f(k)$ denotes the final z-number after k scatterings each of which is denoted by z_k corresponding to the scattering angle θ/k, and θ is very small. Similarly,

$$1 + z_f(n) = (1 + z_n)^n$$

According to Fig. 1 (a single curve),

$$z_n > z_m \qquad \text{if} \quad n < m$$

Assuming (Wolf and Foley, 1989) a linear z–θ relation for small θ, we get

$$z_f(m) > z_f(n) \qquad \text{for} \quad m > n \tag{15}$$

It is interesting to note that if the total scattering angle is θ, which is very small, the limiting z-number as the number of scatterings is infinitely large is given by

$$z = e^{\theta} - 1 \tag{16}$$

This follows from

$$\left(1 + \frac{\theta}{n}\right)^n \to e^\theta \qquad \text{as} \qquad n \to \infty$$

Therefore, the limiting scattering angle θ above which the image will be blurred gives an upper bound of the scattering contribution in the z-number and is given by $e^\theta - 1$. This poses a restriction on the range of values of z-number given by Wolf and James (1990), the upper bound being such that the image will be blurred above this limit.

ACKNOWLEDGMENTS

We are grateful to Prof. Emil Wolf, University of Rochester, for his valuable comments regarding our work and also for his constant inspiration.

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